ISSN 0027-1349, Moscow University Physics Bulletin, 2009, Vol. 64, No. 6, pp. 589–593. © Allerton Press, Inc., 2009.
Original Russian Text © T.G. Elizarova, O.V. Bulatov, 2009, published in Vestnik Moskovskogo Universiteta. Fizika, 2009, No. 6, pp. 29–33.

Numerical Simulation of Gas Flows on the Basis of Quasi-Hydrodynamic Equations

T. G. Elizarova[a, c] and O. V. Bulatov[b]

[a] *Institute of Mathematical Modeling, Russian Academy of Science, Miusskaya pl., 4a, Moscow, 125047 Russia*
[b] *Faculty of Physics Chair of Mathematics, Moscow State University, Moscow, 119991, Russia*
[c] *e-mail: telizar@mail.ru*

Received June 10, 2009, accepted July 8, 2009

Abstract—Based on seven well-known test problems, reflecting the outstanding characteristics of nonstationary flows of an inviscid gas, the possibilities of using a quasi-hydrodynamic algorithm for the numerical solution of Euler equations are investigated.

Key words: quasi-hydrodynamic algorithm, Euler equations, one-dimensional flows

DOI: 10.3103/S002713490906006X

INTRODUCTION

The numerical calculation of nonstationary flows of compressible gas in one dimension by the space approximation is a well-known way to estimate the possibilities of numerical algorithms.

The one-dimensional flows of an ideal gas have been well studied, and for the gap collapse problem type, self-similar solutions of the Euler equations exist [1], [2], which serve as a reliable standard for verification of the accuracy and convergence of numerical solutions. A representative system of tests is collected in [3, 4], where calculations of one-dimensional nonstationary flows of an ideal gas are made on the basis of ten different numerical algorithms. These algorithms sufficiently represent the possibility of the finite-difference approach to solving the equations of gas dynamics in Eulerian variables.

In this paper, these tests verified the possibility of a numerical algorithm based on the quasi-hydrodynamic system of equations constructed in the works of Yu.V. Sheretov (see, for example, [5, 6]).

Quasi-hydrodynamic equations, or Shepetov equations, may be considered as a system describing the flow of a viscous compressible gas and generalizing Navier-Stokes equations. By comparison with the Navier–Stokes system, the Sheretov system involves regularizing additions to the equations in the form of second spatial derivatives with a small parameter as a multiplier. Simplification of these equations is obtained for the case of a viscous incompressible fluid, this allowed us to construct efficient numerical algorithms for solving the nonstationary problems of forced and free convection. However, the use of Sheretov equations to calculate the compressible flows has hardly been studied. The numerical solution of a piston [5], which show the performance of the model, is an exception.

The system of quasi-gas dynamic equations which was the basis for the Sheretov system allowed us to construct a family of numerical algorithms that are extremely efficient for the calculation of compressible viscous gas ([5–7]). However, this system is essentially based on the equation of state of an ideal gas, which is inadequate for many practical applications. It therefore seems timely to explore the possibilities of numerical methods for the computation of compressible flows based on the Sheretov system, for which the equation of state has a more general form.

1. THE SYSTEM OF EQUATIONS AND NUMERICAL ALGORITHM

The quasi-hydrodynamic system of Sheretov equations for one-dimensional plane flows of a viscous gas in conventional notation has the form

$$\frac{\partial \rho}{\partial t} + \frac{\partial j_m}{\partial x} = 0, \quad (1)$$

$$\frac{\partial \rho u}{\partial t} + \frac{\partial j_m u}{\partial x} + \frac{\partial p}{\partial x} = \frac{\partial \Pi}{\partial x}, \quad (2)$$

$$\frac{\partial E}{\partial t} + \frac{\partial j_m H}{\partial x} + \frac{\partial q}{\partial x} = \frac{\partial \Pi u}{\partial x}. \quad (3)$$

Here E and H are the full energy of volume unit and full specific enthalpy that are calculated by the formulas: $E = \rho u^2/2 + \rho\varepsilon$ and $H = (E + p)/\rho$. The mass flux density vector is calculated as

$$j_m = \rho(u - w),$$

where the addition to the speed has the form

$$w = \frac{\tau}{\rho}\left(\rho u\frac{\partial u}{\partial x} + \frac{\partial p}{\partial x}\right).$$

The viscous stress tensor component including in the system of equations (1)–(3) is determined as:

$$\Pi = \frac{4}{3}\mu\frac{\partial u}{\partial x} + \rho u w.$$

The vector of the heat flow q equals

$$q = -\kappa\frac{\partial T}{\partial x},$$

where μ is the coefficient of dynamic viscosity, $\kappa = \mu\gamma R/((\gamma - 1)\mathrm{Pr})$ is the conductivity coefficient, γ is the adiabatic indicator, Pr is the Prandtl number, $\tau = \mu/p\mathrm{Sc}$ is a relaxation parameter having a time dimension, and Sc is the Schmidt number.

The system of equations (1)–(3) allows closure of the form

$$p = p(\rho, T), \quad \varepsilon = \varepsilon(\rho, T). \tag{4}$$

However, in the following calculations for comparison with existing numerical results we will use the state equations of an ideal gas

$$p = \rho RT, \quad \varepsilon = \frac{p}{\rho(\gamma - 1)}.$$

For the convenience of the numerical solution the system of equations (1)–(3) is reduced to a dimensionless form using the basic values of the density ρ_0, sound velocity $c_0 = \sqrt{\gamma R T_0}$ and length L. The reduction to a dimensionless form does not alter the form of the equations.

We introduce a uniform grid to coordinate x with step h and a time grid with step Δt. We will determine the values of all the gasdynamic quantities: velocity, density, and pressure, in the grid nodes. The values of flow values are defined in the half-integer nodes. To solve (1)–(3) we use a scheme that is explicit in time, first proposed in [5] for the numerical solution of the problem of a piston:

$$\hat{\rho}_i = \rho_i - \frac{\Delta t}{h}(j_{mi+1/2} - j_{mi-1/2}), \tag{5}$$

$$\widehat{\rho_i u_i} = \rho_i u_i + \frac{\Delta t}{h}[(\Pi_{i+1/2} - \Pi_{i-1/2}) - (p_{i+1/2} - p_{i-1/2}) - (j_{mi-1/2}u_{i+1/2} - j_{mi-1/2}u_{i-1/2})], \tag{6}$$

$$\hat{E}_i = E_i + \frac{\Delta t}{h}\Bigg[(\Pi_{i+1/2}u_{i+1/2} - \Pi_{i-1/2}u_{i-1/2}) - (q_{i+1/2} - q_{i-1/2}) - \left(\frac{j_{mi+1/2}}{\rho_{i+1/2}}(E_{i+1/2} + p_{i+1/2}) - \frac{j_{mi-1/2}}{\rho_{i-1/2}}(E_{i-1/2} + p_{i-1/2})\right)\Bigg], \tag{7}$$

$$p_i = (\gamma - 1)\left(E_i - \frac{\rho_i u_i^2}{2}\right).$$

The discrete analog of the mass flux density j_m has the form

$$j_{mi+1/2} = \rho_{i+1/2}(u_{i+1/2} - w_{i+1/2}), \tag{8}$$

where addition to the velocity is calculated as

$$w_{i+1/2} = \frac{\tau_{i+1/2}}{\rho_{i+1/2}}\left(\rho_{i+1/2}u_{i+1/2}\frac{u_{i+1} - u_i}{h} + \frac{p_{i+1} - p_i}{h}\right). \tag{9}$$

The discrete expressions for Π and q are written similarly. The order of accuracy of difference scheme (5)–(9) is $O(h^2 + \Delta t)$.

During the numerical solution of Euler equations at the base of the system (1)–(3) all dissipative terms, that is the terms with coefficients μ, κ and τ, are considered as artificial regularizers. Herewith, the relaxation parameter and the coefficients of viscosity and thermal conductivity are interrelated and in dimensionless form are computed as

$$\tau = \alpha\frac{h}{c}, \quad \mu = \tau p\mathrm{Sc}, \quad \kappa = \frac{\tau p\mathrm{Sc}}{\mathrm{Pr}(\gamma - 1)}, \tag{10}$$

where $0 < \alpha$ is the numerical coefficient. In the calculation presented bellow Pr = 1 and Sc = 1.

The scheme (5)–(10) formally has the order $O(\alpha h + \Delta t)$. The calculations presented below confirm that the decrease of coefficient α within certain limits is equivalent to the condensation of the spatial grid by α times.

The difference scheme above (5)–(10) meets the Courant stability condition. The time step is chosen from the ratio

$$\Delta t = \beta_{\min}\left(\frac{h}{c}\right), \tag{11}$$

where $\beta > 0$ is a numerical coefficient or the Courant number.

2. RIEMANN PROBLEMS OF THE COLLAPSE OF DECAY

This section discusses the problem of collapse of decay, as discussed in [3, 4]. These problems are fully reflected in the distinctive and complex features for numerical simulation of nonstationary gas-dynamic flow. The initial data for the collapse of the discontinuity problem are given in Table 1, in accordance with the notation used in [3, 4]. Namely, the values of gasdynamic quantities on the left of the discontinuity are marked by L, on the right, by the index R. The time for which the graphs are built is indicated in Table 1 and is denoted as t_{fin}.

The boundary conditions coincide with the corresponding initial conditions at the ends of the computational domain. In all variants of calculation $\gamma = 1.4$, except for the Noh problem (3), with $\gamma = 5/3$. The length of the domain of calculation is equal to 1, the discontinuity is located at 0. Testing of the quasi-gas dynamic equations in these examples is presented in [8].

The initial conditions for Riemann problems

Test	ρ_L	u_L	p_L	ρ_R	u_R	p_R	$t_{\rm fin}$
1	1	0.75	1	0.125	0	0.1	0.2
2	1	−2	0.4	1	2	0.4	0.15
3	1	1	10^{-6}	1	−1	10^{-6}	1
3a	1	−19.59745	1000	1	−19.59745	0.01	0.012
4	5.99924	19.5975	460.894	5.99924	−6.19633	46.095	0.035
5	1.4	0	1	1	0	1	2
6	1.4	0.1	1	1	0.1	1	2

Test 1. In this problem characteristics features inherent in supersonic flows occur, a wave of rarefaction, contact discontinuity, and a shock wave.

The algorithm considered provides a sustainable solution to this problem as an agreed choice of the coefficients α and β in formulas (10) and (11), respectively. In Fig. 1a the maximum value of the parameter β is shown, which determines the time step for a sustainable account dependence of the value of the regularization parameter α for the test problem 1 and the following examples.

Convergence of numerical solutions to the self-similar solution of the problem (the gray line in this and the following figures) at the concentration of the spatial grid for $\alpha = 0.4$, $\beta = 0.1$ is illustrated in Fig. 1b. Here the value of the regularization parameter α is twice exceeded the corresponding value for the quasi-gas dynamic algorithm [8].

Test 2. In this problem the flow corresponds to two rarefaction waves, scattered from the center of the domain. The complexity of the numerical solution of this problem is due to the fact that in the center between scattered flows the gas density and pressure are very low, but at the same time the internal energy $\varepsilon = p/(\rho(\gamma - 1))$ does not tend to zero. It should be noted that all known finite difference scheme in the Eulerian formulation inadequately describe the behavior of the internal energy in this problem, see, for example, [3, 4].

In Figure 2 the distribution of density and internal energy in this problem for $\alpha = 0.5$, $\beta = 0.1$ is shown. We can see the convergence of solutions to self-similar solution with the concentration of the grid for the graph of density and internal energy. However, in contrast to the results of [8], even for the calculations that are the most detailed in the grid in the distribution of internal energy near the center of domain the maximum is conserved; the existence of this maximum contradicts the self-similar solution and represents the so-called entropy trace. The domain of the sustainable solution of this problem is shown in Fig. 1a.

Tests 3 (Noh problem) and 3a. The first test represents the collision of two hypersonic flows of cold dense gas, which leads to the formation of two scattered infinitely strong shock waves, between which stationary gas with a constant density and pressure remains. Indeed, according to the initial conditions (table), the velocity of sound in the unperturbed flow is $c = \sqrt{\gamma p_R/\rho_R} = 0.0013$. The velocity of the wave propagation is equal to 1, i.e., the Mach number Ma $= u_L/c = 775$. It is known that in terrestrial conditions, the maximum attainable Mach number is around 30. The second test describes the gas-dynamic flow of the type of compression of gas found in a thermonuclear blast. The pressure drop p_L/p_R is 10^5, which corresponds to a temperature drop of the same order.

The difference scheme in (5)–(10) does not allow us to solve these test problems. This implies that the

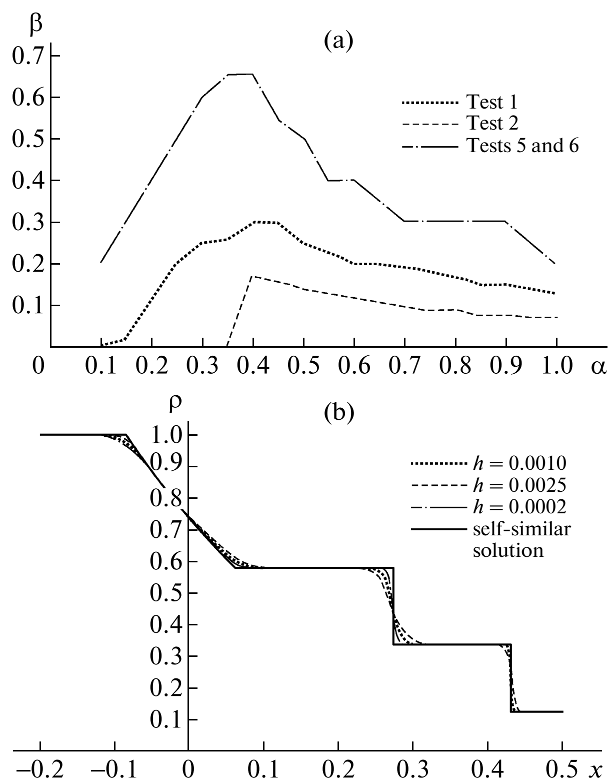

Fig. 1. (a) The condition of algorithm stability for test problems 1, 2 and 5, 6. (b) The distribution of density ρ. The solution dependence of the grid's step, $\alpha = 0.4$, $\beta = 0.1$.

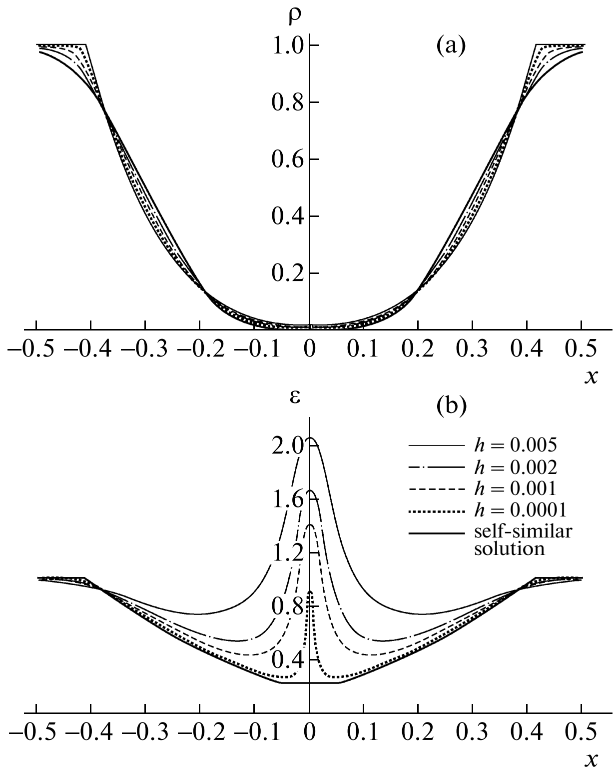

Fig. 2. Test 2. (a) The distribution of density ρ. The convergence of solution on the grid. $\alpha = 0.5$, $\beta = 0.1$. (b) The distribution of internal energy ε. The convergence on the grid. $\alpha = 0.5$, $\beta = 0.1$.

existing regularization in the system of equations is insufficient to smooth the superstrong discontinuity of internal energy arising in the initial moments.

Test 4. In this problem a gas flow in the form of two scattered in gas shock waves is considered; between these waves the moving contact discontinuity is located.

A sustainable solution to this problem is obtained when the parameters of the calculation are $0.3 \leq \alpha \leq 0.8$ and $\beta \leq 0.01$. The convergence of numerical solutions on the grid to self-similar distributions of density and internal energy is shown in Fig. 3. The solution obtained is similar in accuracy to the distributions given in [8], but during its computation an increasingly value of the coefficient $\alpha = 0.7$ is used and this is an order of magnitude smaller than that used in the quasi gas dynamic algorithm, with the Courant number $\beta = 0.01$. Thus, in this test calculation the difference scheme (5)–(10) is less accurate and stable than the difference scheme based on quasi gas dynamic equations.

Test 5. The flow in this problem is a stationary contact discontinuity. At the inflection of the viscosity and thermal conductivity (Sc = 0) the width of the contact discontinuity is one step of the grid. The condition for the stability of this algorithm is shown in Fig. 1a. Figure 4

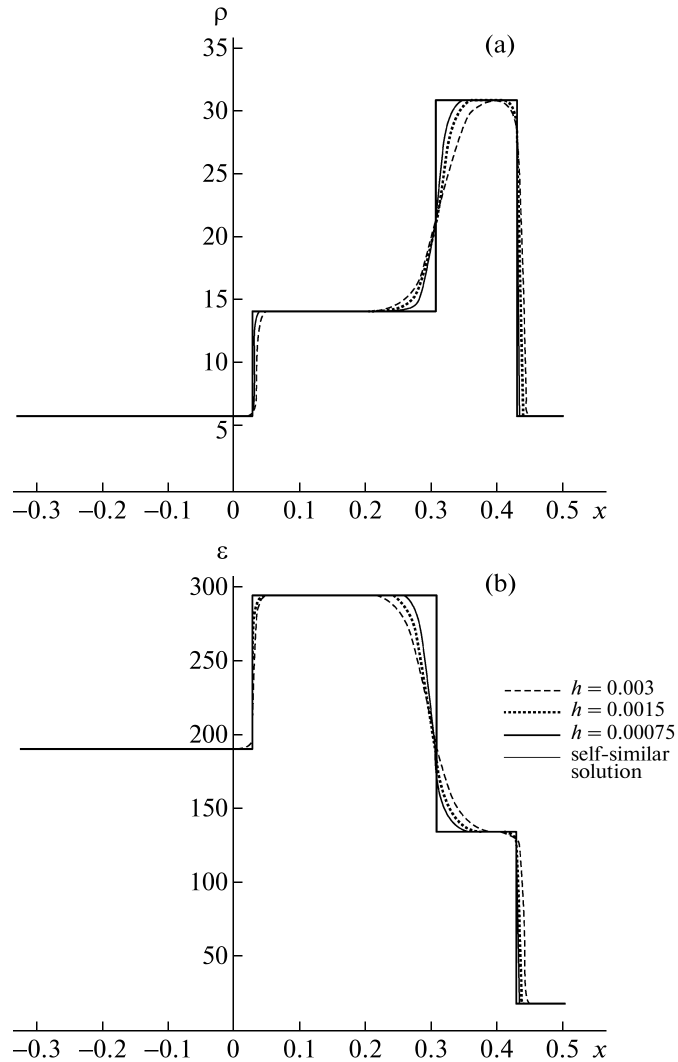

Fig. 3. Test 4. (a) The distribution of density ρ. The convergence on the grid. (b) The distribution of internal energy ε. The convergence on the grid.

shows the density distribution in this problem for Schmidt numbers Sc = 1, 0.1 and 0 for the parameters of calculation $\alpha = 0.5$ and $\beta = 0.5$. For Sc = 0 it also shows the convergence of the numerical solutions on the grid to a self-similar solution.

Test 6. Here, the flow is a contact discontinuity that slowly moving in a gas. The optimal parameters for stable calculation are the same as in the previous test. Unlike the previous case the value Sc = 0 leads to oscillations near the wave front. At Sc = 0.1, as in the problem of stationary contact decay, the width of the discontinuity is one step of the difference grid.

The results of the last two tests are similar in accuracy and stability to the data obtained on the basis of the quasi gas dynamic algorithm [8]. The domain of the stability of the algorithm for both tests is shown in Fig. 1a.

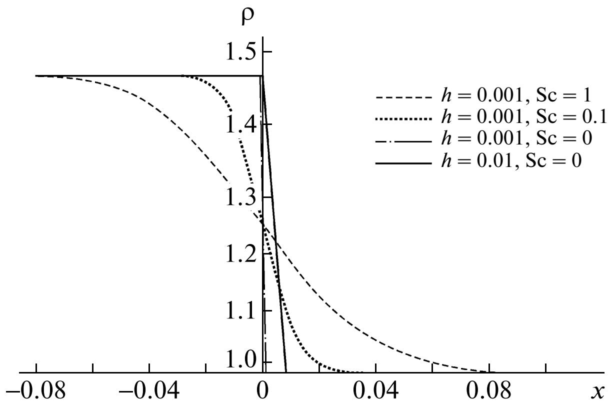

Fig. 4. Test 5. The distribution of density ρ in the stationary contact discontinuity. The solution dependence of the Schmidt number Sc and of a step of the spatial grid.

CONCLUSIONS

The calculations given show that the Sheretov system of quasi-hydrodynamic equations describing the flow of a gas with an equation of type (4) allows uniform calculations, i.e., without the selection of discontinuities, we calculate the nonstationary flows with contact discontinuities and shock waves of low intensity. Performing this numerical algorithm is less stabile and accurate compared to algorithms based on the quasi gas dynamic system of equations that is limited by using of the equation of state of ideal gas. However, the algorithm considered can be widely used in the study of transonic flows encountered in technical applications, for example, flows in chemical reactors, whose numerical simulations are usually based on the equation of state of a nonideal gas.

REFERENCES

1. B. L. Rozhdestvenskii and N. N. Yanenko, *Systems of Quasilinear Equations and Their Applications in Gas Dynamics* (Nauka, Moscow, 1978) [in Russian].
2. A. G. Kulikovskii, N. V. Pogorelov, and A. Yu. Semenov, *Mathematical Aspects of Numerical Solution to Hyperbolic Systems of Equations* (Fizmatlit, Moscow, 2001) [in Russian].
3. R. Liska and B. Wendroff, *Comparison of Several Difference Schemes on 1D and 2D Test Problems for Euler Equations*, Technical Report LA-UR-01-6225 (LANL, Los Alamos, 2001).
4. R. Liska and B. Wendroff, SIAM. J. Sci. Comput. **25**, 995 (2003).
5. Yu. V. Sheretov, *Mathematical Modeling of Liquid and Gas Flows Using Quasi-hydrodynamic and Quasi Gas-Dynamic Equations* (Tver. Gos. Univ., Tver, 2000) [in Russian].
6. T. G. Elizarova, *Quasi Gas Dynamics Equations and Methods of Viscous Flows Calculation* (Nauchnyi Mir, Moscow, 2007) [in Russian].
7. T. G. Elizarova, M. E. Sokolova, and Yu. V. Sheretov, Zh. Vych. Matem. Matem. Fiz. **45**, 544 (2005) [Comput. Math. Math. Phys. **45**, 524–534 (2005)].
8. T. G. Elizarova and E. V. Shil'nikov, Zh. Vych. Matem. Matem. Fiz. **49**, 549 (2009) [Comput. Math. Math. Phys. **49**, 532 (2009)].