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Normalized Shallow Water Equations

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Abstract—A normalized shallow water equation system is constructed based on a normalized Navier–Stokes system and the numerical algorithm of its solution is suggested. The test Riemann problem demonstrates the possibilities of the proposed numerical algorithm.

Key words: normalization, quasi hydrodynamic equations, shallow water equations, numerical method.

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INTRODUCTION

Incompressible liquid flow with a free surface in the gravity field may be described in shallow water approximation (Sen-Venane's equations). The given approximation is constructed in the assumption that the liquid depth is small in comparison with the characteristic dimensions of the problem, such as reservoir bottom asperities. In this case it is possible to neglect the transverse (vertical) velocity component in comparison with velocity components along the layer (horizontal velocity components) and to assume that the transversal velocity component is constant along the layer. In this approximation liquid is described as a medium posessing velocity $\mathbf{u}(x, y, t)$ and layer width $h(x, y, t)$ at point (x, y) at moment in time t.

Corresponding mathematical models are widely used for solving problems of either academical or practical interest. The latter include flow modelling in relatively shallow reservoirs, rivers, water storage areas, flows near oceanfronts, calculation of tsunami waves and water evacuation near hydroelectric power stations and a great number of other problems directly connected with ecological problems. The importance of the mentioned subjects is supported by a large number of scientific articles and conferences devoted to applications of shallow water equations. Methods of obtaining the pointed equation system and its study in different approximations have been discussed, for example, in [1–5]. The study of expressions like shallow water equations, is described in [6].

Shallow water equations by their nature are closely connected with gas dynamics equations and they may be obtained from these equations denoted in Euler approximation under a series of special assumptions. The numerical algorithms for solving shallow water equations are varied and rather complicated. In the given work normalized shallow water equations are suggested, which permit a simple and effective solution method. Solution stability is provided by normalizing additions.

Normalized shallow water equations are constructed based on a new class of gas dynamics equations, which are based on spatio-temporal averaging when defining the main gas gas-dynamic magnitudes of density, velocity and pressure. These equations differ from Euler equations in their additional dissipative components, which serve as normalizers and provide the stability and precision of the corresponding numerical algorithms.

SHALLOW WATER EQUATIONS

An analogy between a shallow water equation system and a Euler equation system, which describes gas-dynamic flows, has been drawn in [2–5, 7]. We will consider this analogy using a simple example.

According to [7], a shallow water equation system for planar one-dimensional flow in the absence of external forces is denoted in the form:

$$\frac{\partial h}{\partial t} + \frac{\partial uh}{\partial x} = 0, \tag{1}$$

$$\frac{\partial hu}{\partial t} + \frac{\partial u^2 h}{\partial x} + \frac{\partial}{\partial x}\left(\frac{gh^2}{2}\right) + gh\frac{\partial b}{\partial x} = 0. \tag{2}$$

The water level height $h(x, t)$ and water velocity $u(x, t)$ have unknown magnitudes. The known magnitude $b(x)$ defines the bottom contour mark and g defines acceleration (Fig. 1).

Let's write out the continuity and pulse equations of the Euler system for planar one-dimensional flow:

$$\frac{\partial \rho}{\partial t} + \frac{\partial \rho u}{\partial x} = 0, \tag{3}$$

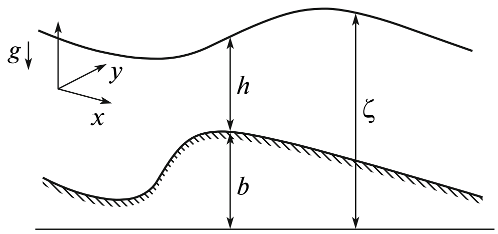

Fig. 1. Designations for shallow water equations.

$$\frac{\partial \rho u}{\partial t} + \frac{\partial \rho u^2}{\partial x} + \frac{\partial p}{\partial x} = 0. \quad (4)$$

Here, the magnitudes of gas density $\rho(x, t)$, velocity $u(x, t)$ and pressure $p(x, t)$ are unknown, which are calculated with the help of state equation. It is obvious that in the set of equations (3)–(4)

$$\rho = h, \quad p = gh^2/2, \quad (5)$$

we obtain shallow water equations (1)–(2) for a flow above a plane, i.e., flows where $b = \text{const}$.

The analogue of the gas-dynamic Mach number $\text{Ma} = u/c$, where $c = \sqrt{\gamma RT}$ is the acoustic speed, in the shallow water model is the Froude number $\text{Fr} = u/c$. At the same time small perturbation propagation velocity is calculated as $c = \sqrt{gh}$.

NORMALIZED SHALLOW WATER EQUATIONS

Let's write out a gas-dynamic equation system taking account of spatio-temporal averaging. As such a system we will use the Sheretov quasi hydrodynamic (QHD) equation system, which is traditionally denoted in the form

$$\frac{\partial \rho}{\partial t} + \text{div}\mathbf{j}_m = 0, \quad (6)$$

$$\frac{\partial(\rho \mathbf{u})}{\partial t} + \text{div}(\mathbf{j}_m \otimes \mathbf{u}) + \nabla p = \text{div}\Pi, \quad (7)$$

$$\frac{\partial}{\partial t}\left[\rho\left(\frac{\mathbf{u}^2}{2} + \varepsilon\right)\right] + \text{div}\left[\mathbf{j}_m\left(\frac{\mathbf{u}^2}{2} + \varepsilon + \frac{p}{\rho}\right)\right] + \text{div}\mathbf{q} = \text{div}(\Pi \cdot \mathbf{u}). \quad (8)$$

Here, density vector of mass flow and viscous stress tensor are calculated as

$$\mathbf{j}_m = \rho(\mathbf{u} - \mathbf{w}), \quad \mathbf{w} = \frac{\tau}{\rho}[\rho(\mathbf{u} \cdot \nabla)\mathbf{u} + \nabla p], \quad (9)$$

$$\Pi = \Pi_{\text{NS}} + \rho \mathbf{u} \otimes \mathbf{w}, \quad (10)$$

where Π_{NS} is the Navier–Stokes viscous stress tensor, $\mathbf{q}$ is the thermal current, and τ is the time dimensioned normalization parameter. Components with the coefficient τ may be considered as normalizing additions to Navier–Stokes equations.

The first two equations of the Sheretov system (6)–(10) for planar one-dimensional flow are denoted in the form

$$\frac{\partial \rho}{\partial t} + \frac{\partial j_m}{\partial x} = 0, \quad (11)$$

$$\frac{\partial \rho u}{\partial t} + \frac{\partial j_m u}{\partial x} + \frac{\partial p}{\partial x} = \frac{\partial \rho u w}{\partial x}, \quad (12)$$

where

$$j_m = \rho(u - w), \quad w = \frac{\tau}{\rho}\left(\rho u \frac{\partial u}{\partial x} + \frac{\partial p}{\partial x}\right). \quad (13)$$

By analogy with the classical approach, making formal substitution (5) in the system (11)–(13) and introducing a component taking account of substrate surface profile, we obtain a shallow water equation system with a normalizer

$$\frac{\partial h}{\partial t} + \frac{\partial j_m}{\partial x} = 0, \quad (14)$$

$$\frac{\partial hu}{\partial t} + \frac{\partial j_m u}{\partial x} + \frac{\partial}{\partial x}\left(\frac{gh^2}{2}\right) + gh\frac{\partial b}{\partial x} = \frac{\partial huw}{\partial x}, \quad (15)$$

where

$$j_m = h(u - w), \quad w = \tau\left(u\frac{\partial u}{\partial x} + g\frac{\partial h}{\partial x} + g\frac{\partial b}{\partial x}\right). \quad (16)$$

Excluding the magnitude j_m (16) in (14)–(15) we have the following equation system

$$\frac{\partial h}{\partial t} + \frac{\partial uh}{\partial x} = \frac{\partial wh}{\partial x}, \quad (17)$$

$$\frac{\partial hu}{\partial t} + \frac{\partial u^2 h}{\partial x} + \frac{\partial}{\partial x}\left(\frac{gh^2}{2}\right) + gh\frac{\partial b}{\partial x} = 2\frac{\partial huw}{\partial x}, \quad (18)$$

where

$$w = \tau\left(u\frac{\partial u}{\partial x} + g\frac{\partial h}{\partial x} + g\frac{\partial b}{\partial x}\right). \quad (19)$$

When $\tau = 0$ the equation system (17)–(19) transforms into the classical shallow water equation system (1), (2). One may show that the additions are on the order of $O(\tau^2)$ for stationary flows. When performing numerical calculations, components with the coefficient τ are considered as normalizing additions.

By analogy with the conclusion found using the equation systems obtained in [7, 13], one may write out a normalized shallow water equation system in the absence of external forces and friction forces for a planar two-dimensional flow. Normalized shallow water equations taking account of external influences (such as wind, Coriolis forces, bottom friction, and other effects) may be obtained if one considers mass and pulse conservation laws for small but finite stationary volume. This method of constructing QHD systems is described in [11, 13].

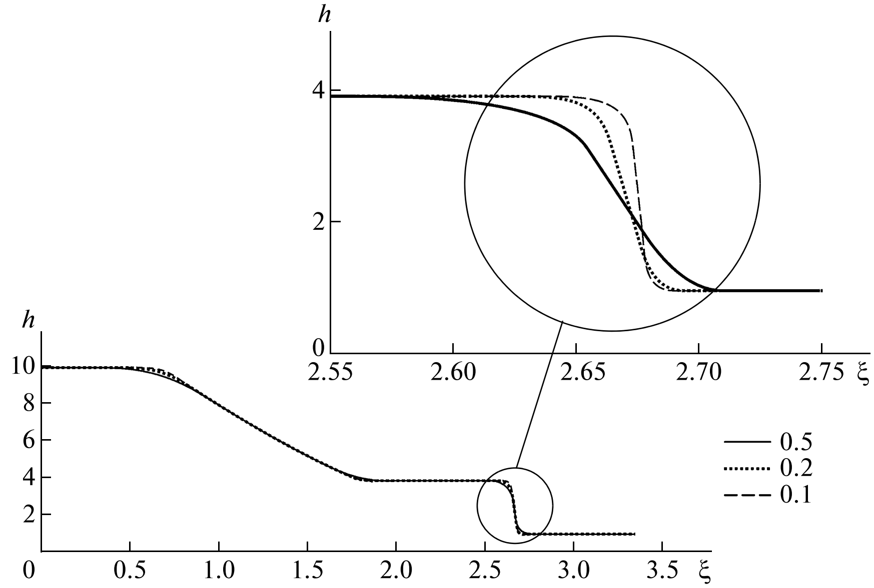

Fig. 2. Convergence by grid, the liquid level height $h(\xi)$. The diagram fragment is shown in an expanded scale.

NUMERICAL ALGORITHM AND CALCULATION RESULTS

A survey of contemporary numerical algorithms for solving shallow water equations is described in [7–10] and in the reference literature.

By analogy with the algorithms elaborated for quasi hydrodynamic equations we will use a time-explicit difference scheme with central difference approximation for all the spatial derivatives for numerical solving normalized shallow water equations. Such difference schemes have high efficiency and accuracy when calculating viscous compressible gas flows (for example, see [11–13]). We will set the values of the decision variables $h(x, y)$ and $u(x, y)$ in the nodes of a spatial grid. Numerical algorithm stability is provided by components with the coefficient τ, whose magnitude is connected with the spatial grid step h_x and is calculated as

$$\tau = \alpha\frac{h_x}{c}, \quad c = \sqrt{gh}, \tag{20}$$

where $0 < \alpha < 1$ is the numerical coefficient selected based on accuracy conditions and calculation stability. The stability condition is of the form of the Courant condition, where the time step is selected as

$$\Delta t = \beta h_x/c. \tag{21}$$

Here the coefficient $0 < \beta < 1$ depends on the magnitude of the normalization parameter τ and is selected in the calculation process in order to provide numerical solution consistency.

To check the method's efficiency we chose solution of the Rieman problem, which is an initial-boundary breach collapse problem. This problem was applied in [7, 9] to test numerical methods of solving shallow water equations based on difference algorithms such as Godunov schemes of the first and second orders of accuracy.

Let's consider planar one-dimensional liquid flow in a channel with the length L and flat bottom b = const. At the initial moment of time a liquid level breach occurs in the center of the domain so that it separates two homogeneous states with the level height $h = h_l$ on the left and $h = h_r$ on the right. At the initial moment of time the liquid is stationary on the both sides of the breach, $u_l = u_r = 0$. According to [7, 9] L = 100 m, h_l =10 m, h_r = 1 m. The calculation continues until the moment of time T_f = 2 s.

Uniform spatial grids were used in the calculations. For convenience, the equations were reduced to non-dimensional form using the magnitudes h_r and $\sqrt{gh_r}$, where $g = 9.8$ m/s^2.

Figure 2 shows the liquid height profiles $h(x)$ obtained on a sequence of uniform spatial grids with the steps h_x = 0.1, 0.2, and 0.5; the normalization parameter (20) is calculated at $\alpha = 0.5$, and the time step (21) corresponds to $\beta = 0.3$. For comparison with the data from [7] the results are shown in the form of $h(\xi)$ dependence, where $\zeta = x/(t\sqrt{gh_l})$ is a self-simulated variable. The diagrams show rapid and monotonous convergence of the numerical solution to the self-simulated solution when the spatial grid thickens. Figure 3 shows the distribution of velocity $u(x)$ on the grid $h_x = 0.5$.

The stability of the numerical solution is provided by normalizing τ-additions. When calculating normalization parameter τ (Eq. (20)) there is one free parameter, the numerical coefficient $0 < \alpha < 1$. Figure 4 demonstrates the dependence of the time-step (the

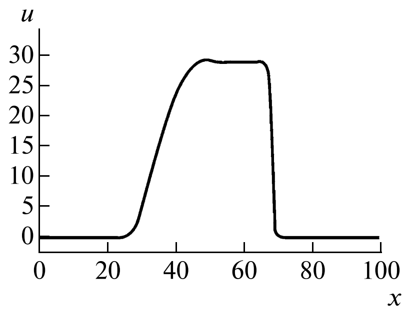

Fig. 3. Distributions of velocity $u(x)$, calculation on the grid with step $h_x = 0.5$.

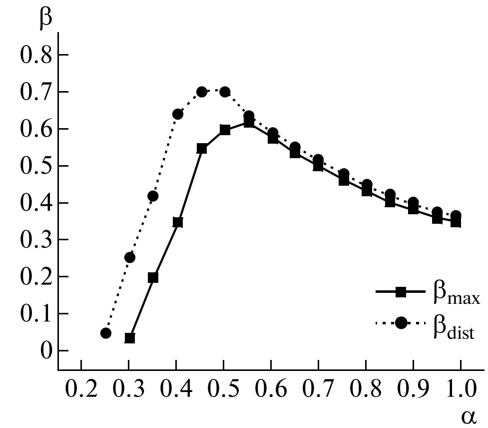

Fig. 4. Method stability dependence on the normalization parameter magnitude.

coefficient β) on the normalization parameter (the coefficient α), which provides a stable solution of the problem. Here, β_{max} corresponds to the case where velocity profile oscillations constitute 1–3% of the velocity magnitude at the corresponding point, the points β_{dist} correspond to the 10–12% criterion. It is necessary to note that velocity $u(x, t)$ in the considered problem is a more sensitive characteristic than liquid layer height $h(x, t)$.

The stability data obtained for the grids $h_x = 0.1$, 0.2, and 0.5 practically coincide, confirming that Courant conditions (20) are true for the considered algorithm and testifying to the existence of an optimal normalization parameter value corresponding to $\alpha \sim 0.5$. Further increase of the normalization parameter magnitude leads to a decreasing of the scheme stability. At the same time the values β_{max} and β_{dist} in this region get closer and at $\alpha > 0.6$ they practically coincide; this shows that dependence of the solution on the magnitude τ becomes stricter.

According to comparison of the obtained results with the data from [7, 9] the suggested algorithm is less precise than Godunov scheme methods for rather course grids. However, the algorithm accuracy increases when the spatial grids broaden and the calculation expenses are compensated by algorithmic simplicity and high calculation efficiency. The latter is very important in connection with contemporary multiprocessor and multinuclear calculation modules.

CONCLUSIONS

The normalized shallow water equations and an effective numerical solution algorithm were constructed in this given work. The method convergence and accuracy was demonstrated on the example of classical Rieman problem solution.

In contrast to known methods using complicated and expensive solution algorithms for solving shallow water equations [7–10] the suggested method seems to be the simplest from the algorithmic realization viewpoint. Therefore, the given method is naturally generalized on non-structured grids and it may be effectively realized on multiprocessor calculation systems. This will allow the modeling of flows corresponding to real natural conditions, such as flood flows, river overflows, flows in hydrosystems, precipitation tanks, rivers and ocean foreshores.

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