ISSN 1064-5624, Doklady Mathematics, 2014, Vol. 90, No. 3, pp. 719–723. © Pleiades Publishing, Ltd., 2014.
Original Russian Text © T.G. Elizarova, A.A. Zlotnik, B.N. Chetverushkin, 2014, published in Doklady Akademii Nauk, 2014, Vol. 459, No. 4, pp. 395–399.

MATHEMATICS

On Quasi-Gasdynamic and Quasi-Hydrodynamic Equations for Binary Gas Mixtures

T. G. Elizarova[a], A. A. Zlotnik[b, c], and Academician B. N. Chetverushkin[a]

Received July 9, 2014

DOI: 10.1134/S1064562414070217

The quasi-gasdynamic (QGD) approach makes it possible to construct convenient and reliable difference schemes for the numerical solution of various gasdynamic problems. Its description can be found in [1–3]. More specifically, in [2, Chapter 9] (see also [4]), the Boltzmann kinetic equation for a mixture of monatomic gases [5] is used to derive and test QGD equations for binary mixtures of nonreactive ideal polytropic gases.

In this paper, we analyze and expand the capabilities of the QGD approach in this area. The original equations from [2] are rewritten as conservation laws, which are more conventional in viscous gas dynamics and convenient for discretization. Additionally, an external force and a heat source are taken into account. We briefly discuss the parabolicity of the system in the sense of Petrovskii, which ensures that the system is well defined. An entropy balance equation is derived, and the entropy production for a gas mixture is shown to be nonnegative, which ensures that the system is physically consistent (but does not hold in all available descriptions of gas mixtures). Importantly, to achieve the latter property, the expressions for the exchange terms in the total energy balance equation (initially derived only for monatomic gas mixtures) are properly generalized. Additionally, we introduce a simplification of the QGD system for binary mixtures, which is referred to as a quasi-hydrodynamic system and is used for the numerical simulation of weakly compressible sub- and transonic flows. At the end of this paper, we present simplified barotropic versions of both systems and derive a corresponding energy balance equation with nonpositive energy production.

[a] *Institute of Applied Mathematics, Russian Academy of Sciences, Miusskaya pl. 4, Moscow, 125047 Russia*
[b] *National Research University Higher School of Economics, Myasnitskaya ul. 20, Moscow, 101000 Russia*
[c] *Moscow Power Engineering Institute (Technical University), Krasnokazarmennaya ul. 14, Moscow, 111250 Russia*
e-mail: telizar@mail.ru; azlotnik2007@mail.ru; chetver@imamod.ru

The QGD system for binary gas mixtures a and b (see [2]) consists of the following mass balance, momentum, and total energy equations for the gas α:

$$\partial_t \rho_\alpha + \operatorname{div}(\rho_\alpha \mathbf{u}_\alpha) = \operatorname{div}\{\tau[\operatorname{div}(\rho_\alpha \mathbf{u}_\alpha \otimes \mathbf{u}_\alpha) + \nabla p_\alpha]\}, \quad (1)$$

$$\partial_t(\rho_\alpha \mathbf{u}_\alpha) + \operatorname{div}(\rho_\alpha \mathbf{u}_\alpha \otimes \mathbf{u}_\alpha) + \nabla p_\alpha = \operatorname{div}\{\tau[\operatorname{div}(\rho_\alpha \mathbf{u}_\alpha \otimes \mathbf{u}_\alpha \otimes \mathbf{u}_\alpha) + \nabla \otimes p_\alpha \mathbf{u}_\alpha + (\nabla \otimes p_\alpha \mathbf{u}_\alpha)^{\mathrm{T}}]\} + \nabla\{\tau[\operatorname{div}(p_\alpha \mathbf{u}_\alpha)]\} + \mathbf{S}_{u,\alpha}, \quad (2)$$

$$\partial_t E_\alpha + \operatorname{div}[(E_\alpha + p_\alpha)\mathbf{u}_\alpha] = \operatorname{div}\left\{\tau\left[\operatorname{div}((E_\alpha + 2p_\alpha)\mathbf{u}_\alpha \otimes \mathbf{u}_\alpha) + \nabla\left(\frac{1}{2}|\mathbf{u}_\alpha|^2 p_\alpha\right) + \frac{\gamma_\alpha}{\gamma_\alpha - 1}\left(\frac{p_\alpha}{\rho_\alpha}\nabla p_\alpha + p_\alpha \nabla \frac{p_\alpha}{\rho_\alpha}\right)\right]\right\} + S_{E,\alpha}, \quad (3)$$

where $\alpha = a, b$. Here and below, ∂_t and ∂_i denote the partial derivatives with respect to t and x_i, the operators div and ∇ are taken with respect to x (the divergence of a tensor is taken over its first index), and $\otimes$ denotes the tensor product of vectors.

The basic sought functions are the density $\rho_\alpha > 0$, velocity $\mathbf{u}_\alpha = (u_{1\alpha}, \ldots, u_{n\alpha})$, and absolute temperature $\theta_\alpha > 0$ of the gas α, which are functions of time t and the coordinates $x = (x_1, \ldots, x_n)$, where $n = 1, 2, 3$. The equations also involve the total energy, pressure, and specific internal energy

$$E_\alpha = \frac{1}{2}\rho_\alpha |\mathbf{u}_\alpha|^2 + \rho_\alpha \varepsilon_\alpha, \quad p_\alpha = R_\alpha \rho_\alpha \theta_\alpha, \quad \varepsilon_\alpha = c_{V_\alpha}\theta_\alpha,$$

of the gas α, which is assumed to be ideal polytropic. Here, $R_\alpha = \dfrac{R_0}{m_\alpha}$ (where R_0 is the universal gas constant and m_α is the molecular weight) and the specific heat capacity at constant volume c_{V_α} are positive constants.

Additionally, we use the expression $p_\alpha = (\gamma_\alpha - 1)\rho_\alpha\varepsilon_\alpha$, where $\gamma_\alpha - 1 = \dfrac{R_\alpha}{c_{V_\alpha}}$.

Moreover, $\tau = \tau(\rho_a, \rho_b, \theta_a, \theta_b) > 0$ is the relaxation parameter, and $\mathbf{S}_{u,\alpha}$ and $S_{E,\alpha}$ are exchange terms relating the equations for the gases a and b (their expressions will be given below).

As was noted above, the QGD system (1)–(3) was obtained only for monatomic gas mixtures, i.e., for $\gamma_a = \gamma_b = \dfrac{5}{3}$, and then was formally extended to the case of any $\gamma_\alpha > 1$ and $\gamma_\beta > 1$. Note that the right-hand side of Eq. (3) can be written more compactly by using the formula

$$\nabla\left(\frac{1}{2}|\mathbf{u}_\alpha|^2 p_\alpha\right) + \frac{\gamma_\alpha}{\gamma_\alpha - 1}\left(\frac{p_\alpha}{\rho_a}\nabla p_\alpha + p_\alpha\nabla\frac{p_\alpha}{\rho_a}\right) = \nabla\left(\frac{p_\alpha}{\rho_\alpha}(E_\alpha + p_\alpha)\right).$$

As was recommended in [2] (by analogy with the case of a single gas), the multiplier $\alpha_{\mathrm{Pr}}^{-1}$ preceding the term $p_\alpha\nabla\left(\dfrac{p_\alpha}{\rho_\alpha}\right)$ is introduced into the right-hand side of Eq. (3). Here, $\alpha_{\mathrm{Pr}} > 0$ is the Prandtl number (for $\alpha_{\mathrm{Pr}} = 1$, the multiplier is not necessary).

According to [2, Subsection 3.3], Eqs. (1)–(3) can be rewritten in the following form, which is more conventional in viscous gas dynamics and convenient for discretization:

$$\partial_t\rho_\alpha + \mathrm{div}[\rho_\alpha(\mathbf{u}_\alpha - \mathbf{w}_\alpha)] = 0, \tag{4}$$

$$\partial_t(\rho_\alpha\mathbf{u}_\alpha) + \mathrm{div}[\rho_a(\mathbf{u}_\alpha - \mathbf{w}_\alpha)\otimes\mathbf{u}_\alpha] + \nabla p_\alpha = \mathrm{div}\Pi_\alpha + [\rho_\alpha - \tau\mathrm{div}(\rho_\alpha\mathbf{u}_\alpha)]\mathbf{F}_\alpha + \mathbf{S}_{u,\alpha}, \tag{5}$$

$$\partial_t E_\alpha + \mathrm{div}[(E_\alpha + p_\alpha)(\mathbf{u}_\alpha - \mathbf{w}_\alpha)] = \mathrm{div}(-\mathbf{q}_\alpha + \Pi_\alpha\mathbf{u}_\alpha) + \rho_\alpha(\mathbf{u}_\alpha - \mathbf{w}_\alpha)\cdot\mathbf{F}_\alpha + Q_\alpha + S_{E,\alpha} \tag{6}$$

with $\mathbf{F}_\alpha = 0$ and $Q_\alpha = 0$. Here, following a well-known method from QGD theory [2, 3], we add given densities of body force $\mathbf{F}_\alpha$ and heat source $Q_\alpha \ge 0$ to analyze the system in more generality. The sign $\cdot$ denotes the scalar product of vectors.

In these equations, the viscous stress tensor Π_α is given by

$$\Pi_\alpha = \Pi_{NS\alpha} + \rho_\alpha\mathbf{u}_\alpha\otimes\hat{\mathbf{w}}_\alpha + \tau[(\mathbf{u}_\alpha\cdot\nabla)p_\alpha + \gamma_\alpha p_\alpha\mathrm{div}\mathbf{u}_\alpha - (\gamma_\alpha - 1)Q_\alpha]\mathbb{I}, \tag{7}$$

where $\Pi_{NS\alpha}$ is the classical Navier–Stokes viscous stress tensor

$$\Pi_{NS\alpha} = \mu_\alpha\left[2\mathbb{D}(\mathbf{u}_\alpha) - \frac{2}{3}(\mathrm{div}\mathbf{u}_\alpha)\mathbb{I}\right] + \lambda_\alpha(\mathrm{div}\mathbf{u}_\alpha)\mathbb{I},$$

$$\mathbb{D}_{ij}(\mathbf{u}_\alpha) = \frac{1}{2}(\partial_i u_{j\alpha} + \partial_j u_{i\alpha})$$

with dynamic and bulk viscosity coefficients of the form

$$\mu_\alpha = \tau p_\alpha, \quad \lambda_\alpha = \tau p_\alpha\left(\frac{5}{3} - \gamma_\alpha\right), \tag{8}$$

and $\mathbb{I}$ is the unit tensor (of order n). (Note that $\lambda_\alpha = 0$ is obtained if the terms

$$\nabla\left[\tau p_\alpha\left(\gamma_\alpha - \frac{5}{3}\right)(\mathrm{div}\mathbf{u}_\alpha)\mathbb{I}\right],$$

$$\mathrm{div}\left[\tau p_\alpha\left(\gamma_\alpha - \frac{5}{3}\right)(\mathrm{div}\mathbf{u}_\alpha)\mathbf{u}_\alpha\right]\Big)$$

are added to the right-hand sides of Eqs. (2) and (3) in the case $\gamma_\alpha \ne \dfrac{5}{3}$, respectively.)

The auxiliary vectors $\mathbf{w}_\alpha$ and $\hat{\mathbf{w}}_\alpha$ are defined as

$$\mathbf{w}_\alpha = \frac{\tau}{\rho_\alpha}[\mathrm{div}(\rho_\alpha\mathbf{u}_\alpha\otimes\mathbf{u}_\alpha) + \nabla p_\alpha - \rho_\alpha\mathbf{F}_\alpha],$$

$$\hat{\mathbf{w}}_\alpha = \frac{\tau}{\rho_\alpha}[\rho_\alpha(\mathbf{u}_\alpha\cdot\nabla)\mathbf{u}_\alpha + \nabla p_\alpha - \rho_\alpha\mathbf{F}_\alpha],$$

where $\rho_\alpha(\mathbf{u}_\alpha - \mathbf{w}_\alpha)$ is the mass flux density of the gas α.

The heat flux $\mathbf{q}_\alpha$ is given by the formula

$$-\mathbf{q}_\alpha = \kappa_\alpha\nabla\theta_\alpha + \tau\left[\rho_\alpha\left((\mathbf{u}_\alpha\cdot\nabla)\varepsilon_\alpha - \frac{p_\alpha}{\rho_\alpha^2}(\mathbf{u}_\alpha\cdot\nabla)\rho_\alpha\right) - Q_\alpha\right]\mathbf{u}_\alpha$$

with the thermal conductivity

$$\kappa_\alpha = \frac{\gamma_\alpha c_{V_\alpha}}{\alpha_{\mathrm{Pr}}}\tau p_\alpha. \tag{9}$$

Below, formulas (8) and (9) are replaced by the general dependences

$$\mu_\alpha = \mu_\alpha(\rho_\alpha, \theta_\alpha) > 0, \quad \lambda_\alpha = \lambda_\alpha(\rho_\alpha, \theta_\alpha) \ge 0,$$

$$\kappa_\alpha = \kappa_\alpha(\rho_\alpha, \theta_\alpha) > 0.$$

Importantly, since the exchange terms usually depend only on the sought functions (but are independent of their highest derivatives with respect to x and t), the results of [6] that the QGD system for a single gas is Petrovskii parabolic can obviously be extended to QGD system (4)–(6) (in the corresponding analysis, the system splits into nearly independent auxiliary subsystems for a and b, which are related only in terms of τ). These results ensure that the system is well posed mathematically.

The entropy of the gas α and the entropy of the mixture are given by the formulas

$$s_\alpha = S_0 - (\gamma_\alpha - 1)c_{V_\alpha}\ln\rho_\alpha + c_{V_\alpha}\ln\varepsilon_\alpha,$$

$$s = \frac{\rho_a s_a + \rho_b s_b}{\rho}, \quad \rho = \rho_a + \rho_b,$$

where S_0 is a constant.

Theorem 1. *The following entropy balance equation holds for gas mixtures:*

$$\partial_t(\rho s) + \operatorname{div}[\rho_a s_a(\mathbf{u}_a - \mathbf{w}_a) + \rho_b s_b(\mathbf{u}_b - \mathbf{w}_b)] = -\operatorname{div}\left(\frac{\mathbf{q}_a}{\theta_a} + \frac{\mathbf{q}_b}{\theta_b}\right) + \mathcal{P}_s, \tag{10}$$

in which the entropy production $\mathcal{P}_s$ is given by the formulas

$$\mathcal{P}_s := \frac{1}{\theta_a}\Xi_a + \frac{1}{\theta_b}\Xi_b + \frac{1}{\theta_a}(S_{E,a} - \mathbf{S}_{u,a}\cdot\mathbf{u}_a) + \frac{1}{\theta_b}(S_{E,b} - \mathbf{S}_{u,b}\cdot\mathbf{u}_b),$$

$$\Xi_\alpha = \Xi_{NS,1\alpha} + \frac{\kappa_\alpha}{\theta_\alpha}|\nabla\theta_\alpha|^2 + \frac{\rho_\alpha}{\tau}|\hat{\mathbf{w}}_\alpha|^2 + \frac{\tau R_\alpha\theta_\alpha}{\rho_\alpha}[\operatorname{div}(\rho_\alpha\mathbf{u}_\alpha)]^2 + \frac{\tau c_{V_\alpha}\rho_\alpha}{\theta_\alpha}$$
$$\times\left[(\gamma_\alpha - 1)\theta_\alpha\operatorname{div}\mathbf{u}_\alpha + (\mathbf{u}_\alpha\cdot\nabla)\theta_\alpha - \frac{Q_\alpha}{2c_{V_\alpha}\rho_\alpha}\right]^2 + Q_\alpha\left[1 - \frac{\tau(\gamma_\alpha - 1)Q_\alpha}{4p_\alpha}\right], \tag{11}$$

$$\Xi_{NS,1\alpha} = 2\mu_\alpha|\mathbb{D}(\mathbf{u}_\alpha)|^2 + \left(\lambda_\alpha - \frac{2}{3}\mu_\alpha\right)(\operatorname{div}\mathbf{u}_\alpha)^2 \geq 0, \tag{12}$$

where $\alpha = a, b$ and $|\mathbb{D}(\mathbf{u}_\alpha)|^2$ is the square of the length $\mathbb{D}(\mathbf{u}_\alpha)$ (as an n^2-dimensional vector).

If $\tau(\gamma_\alpha - 1)\dfrac{Q_\alpha}{4p_\alpha} \leq 1$ *with* $\alpha = a, b$ *and*

$$\frac{1}{\theta_a}(S_{E,a} - \mathbf{S}_{u,a}\cdot\mathbf{u}_a) + \frac{1}{\theta_b}(S_{E,b} - \mathbf{S}_{u,b}\cdot\mathbf{u}_b) \geq 0 \tag{13}$$

then $\mathcal{P}_s \geq 0$; i.e., the entropy production is nonnegative.

To prove this result, we consider the entropy balance equation for the gas α:

$$\partial_t(\rho_\alpha s_\alpha) + \operatorname{div}[\rho_\alpha s_\alpha(\mathbf{u}_\alpha - \mathbf{w}_\alpha)] = -\operatorname{div}\frac{\mathbf{q}_\alpha}{\theta_\alpha} + \frac{1}{\theta_\alpha}\Xi_\alpha + \frac{1}{\theta_\alpha}(S_{E,\alpha} - \mathbf{S}_{u,\alpha}\cdot\mathbf{u}_\alpha). \tag{14}$$

It is derived by multiplying momentum equation (5) by $\mathbf{u}_\alpha$, subtracting it from the total energy balance equation (6), and dividing the result by θ_α. From this, we obtain the last (exchange) term on the right-hand side of (14). Formulas (11) and (12) with $Q_\alpha = 0$ are derived in [2, 3]. A more concise derivation in a more general situation (for the equations of state of a real gas with $Q_\alpha \neq 0$) is presented in [7]. Another form of these formulas can also be found in [7]. Summing Eqs. (14) over $\alpha = a, b$, we obtain Eq. (10).

The standard exchange terms used in the kinetic theory of gases (and applied in [2]) have the form [5]

$$\mathbf{S}_{u,\alpha} = \nu_{\alpha\beta}\rho_a(\hat{\mathbf{u}} - \mathbf{u}_\alpha), \quad S_{E,\alpha} = \nu_{\alpha\beta}(\hat{E}_\alpha - E_\alpha),$$

where $\nu_{\alpha\beta} > 0$ is the collision frequency of gas molecules α with gas molecules β; here, $\beta = b$ for $\alpha = a$ or $\beta = a$ for $\alpha = b$. Moreover,

$$\hat{\mathbf{u}} = \frac{m_a\mathbf{u}_a + m_b\mathbf{u}_b}{m_a + m_b},$$
$$\hat{E}_\alpha = \frac{1}{2}\rho_\alpha|\hat{\mathbf{u}}|^2 + c_{V_\alpha}\rho_\alpha\hat{\theta}_\alpha, \tag{15}$$
$$\hat{\theta}_\alpha = \theta_\alpha + C_m\left[\theta_\beta - \theta_\alpha + \frac{(\gamma_\alpha - 1)m_\beta}{4R_0}|\mathbf{u}_a - \mathbf{u}_b|^2\right]$$

with $C_m = \dfrac{2m_am_b}{(m_a + m_b)^2}$. Here, the multiplier $\dfrac{(\gamma_\alpha - 1)m_\beta}{4R_0}$ is generalized as compared with [5] (where only the case $\gamma_a = \gamma_b = \dfrac{5}{3}$ was considered) so that the balance equality holds for the exchange terms in the total energy equation (see (18) below) (unfortunately, this point was overlooked in some works, for example, in [2, 8]). Note the equality

$$C_m m_\beta|\mathbf{u}_a - \mathbf{u}_b|^2 = 2m_\alpha|\hat{\mathbf{u}} - \mathbf{u}_\alpha|^2. \tag{16}$$

Below, we repeatedly use the balance equality

$$\frac{\rho_a}{m_a}\nu_{ab} = \frac{\rho_b}{m_b}\nu_{ba} = N_{\text{col}},$$

where N_{col} is the total number of collisions between gas molecules a and b.

In view of (16), for the above exchange terms, we have

$$\frac{1}{\theta_a}(S_{E,a} - \mathbf{S}_{u,a}\cdot\mathbf{u}_a) + \frac{1}{\theta_b}(S_{E,b} - \mathbf{S}_{u,b}\cdot\mathbf{u}_b)$$
$$= \sum_{\alpha = a,b}\nu_{\alpha\beta}\left[\frac{1}{2}\frac{\rho_\alpha}{\theta_\alpha}(|\hat{\mathbf{u}}|^2 - |\mathbf{u}_\alpha|^2 - (2\hat{\mathbf{u}}\cdot\mathbf{u}_\alpha - |\mathbf{u}_\alpha|^2)) + c_{V_\alpha}\rho_\alpha\left(\frac{\hat{\theta}_\alpha}{\theta_\alpha} - 1\right)\right] = \sum_{\alpha = a,b}\nu_{\alpha\beta}\left[\frac{1}{2}\frac{\rho_\alpha}{\theta_\alpha}|\hat{\mathbf{u}} - \mathbf{u}_\alpha|^2\right.$$
$$\left. + c_{V_\alpha}\rho_\alpha C_m\left(\frac{\theta_\beta}{\theta_\alpha} - 1\right) + \frac{c_{V_\alpha}\rho_\alpha m_\alpha(\gamma_\alpha - 1)}{2R_0\theta_\alpha}|\hat{\mathbf{u}} - \mathbf{u}_\alpha|^2\right]$$
$$= N_{\text{col}}\sum_{\alpha = a,b}\frac{m_\alpha}{\theta_\alpha}|\hat{\mathbf{u}} - \mathbf{u}_\alpha|^2 + N_{\text{col}}C_m\left[m_ac_{V_a}\left(\frac{\theta_b}{\theta_a} - 1\right) + m_bc_{V_b}\left(\frac{\theta_a}{\theta_b} - 1\right)\right].$$

If $m_ac_{V_a} = m_bc_{V_b}$ or, equivalently, $\gamma_a = \gamma_b$, then the expression in square brackets is nonnegative (since $\xi + \xi^{-1} \geq 2$ for $\xi > 0$). Thus, condition (13) holds and the entropy production is nonnegative. This means that the considered system is physically consistent.

Recall the following properties of the exchange terms:

$$\mathbf{S}_{u,a} + \mathbf{S}_{u,b} = 0, \quad \mathbf{S}_{u,a} \cdot \mathbf{u}_a + \mathbf{S}_{u,b} \cdot \mathbf{u}_b = -N_{\text{col}} \frac{m_a m_b}{m_a + m_b} |\mathbf{u}_a - \mathbf{u}_b|^2 \le 0. \tag{17}$$

Moreover, under the same condition $\gamma_a = \gamma_b$,we have another balance equality

$$S_{E,a} + S_{E,b} = N_{\text{col}} \Big[\frac{1}{2}(m_a + m_b)|\hat{\mathbf{u}}|^2 + \sum_{\alpha = a,b} m_\alpha c_{V_\alpha}(\hat{\theta}_\alpha - \theta_\alpha) - \frac{1}{2} m_\alpha |\mathbf{u}_\alpha|^2 \Big]$$
$$= N_{\text{col}} \Big[-\frac{m_a m_b}{2(m_a + m_b)} |\mathbf{u}_a - \mathbf{u}_b|^2 + C_m \sum_{\alpha = a,b} m_\alpha c_{V_\alpha}(\theta_\beta - \theta_\alpha) + C_m \frac{m_a + m_b}{4} |\mathbf{u}_a - \mathbf{u}_b|^2 \Big] = 0. \tag{18}$$

Importantly, a further modification of the second expression in (15),

$$\hat{\theta}_\alpha = \theta_\alpha + C_{m\alpha}(\theta_\beta - \theta_\alpha) + C_m \frac{(\gamma_\alpha - 1) m_\beta}{4 R_0} |\mathbf{u}_a - \mathbf{u}_b|^2,$$
$$C_{m\alpha} := \frac{3}{2}(\gamma_\alpha - 1) C_m$$

ensures the property $m_a c_{V_a} C_{ma} = m_b c_{V_b} C_{mb}$ (irrespective of the explicit expression for C_m in the formula for $C_{m\alpha}$). Then, for any γ_a and γ_b (rather than only for equal ones), we have balance equality (18) and inequality (13), which guarantee that the entropy production is nonnegative. The modifications proposed deserve a further physical analysis.

The QGD system of equations describes gas flows in the entire range of velocities. In [1–3] a simplified quasi-hydrodynamic system was presented for weakly compressible sub- and transonic flows. Performing similar simplifications in QGD system (4)–(6), we obtain a quasi-hydrodynamic system of equations for binary mixtures:

$$\partial_t \rho_\alpha + \operatorname{div}[\rho_\alpha(\mathbf{u}_\alpha - \hat{\mathbf{w}}_\alpha)] = 0, \tag{19}$$

$$\partial_t(\rho_\alpha \mathbf{u}_\alpha) + \operatorname{div}[\rho_\alpha(\mathbf{u}_\alpha - \hat{\mathbf{w}}_\alpha) \otimes \mathbf{u}_\alpha] + \nabla p_\alpha = \operatorname{div}\hat{\Pi}_\alpha + \rho_\alpha \mathbf{F}_\alpha + \mathbf{S}_{u,\alpha}, \tag{20}$$

$$\partial_t E_\alpha + \operatorname{div}[(E_\alpha + p_\alpha)(\mathbf{u}_\alpha - \hat{\mathbf{w}}_\alpha)] = \operatorname{div}(-\hat{\mathbf{q}}_\alpha + \hat{\Pi}_\alpha \mathbf{u}_\alpha) + \rho_\alpha(\mathbf{u}_\alpha - \hat{\mathbf{w}}_\alpha) \cdot \mathbf{F}_\alpha + Q_\alpha + S_{E,\alpha}, \tag{21}$$

where $\hat{\Pi}_\alpha = \Pi_{NS\alpha} + \rho_\alpha \mathbf{u}_\alpha \otimes \hat{\mathbf{w}}_\alpha$, $-\hat{\mathbf{q}}_\alpha = \kappa_\alpha \nabla \theta_\alpha$, and $\alpha = a, b$. This system cannot be directly derived from the Boltzmann equation. The Petrovskii parabolicity of this system obviously follows from [9]. Theorem 1 also remains valid for it with the only difference being that the expression for Ξ_α considerably simplifies to become $\Xi_\alpha = \Xi_{NS,1\alpha} + \frac{\rho_\alpha}{\tau}|\hat{\mathbf{w}}_\alpha|^2 + Q_\alpha$, while the condition on τ (connected with Q_α) is dropped.

In some applications, the systems represented can be further simplified. Specifically, this is associated with passing to the barotropic (in particular, isothermal or adiabatic) case and considering only the mass and momentum balance equations (4), (5) or (19), (20) supplemented with an equation of state $p_\alpha = p_\alpha(\rho_\alpha)$, specifically, $p_\alpha(\rho_\alpha) = p_{1\alpha}\rho_\alpha^{\gamma_\alpha}$ with $p_{1\alpha} > 0$ and $\gamma_\alpha \ge 1$. In the general barotropic case, the coefficient γ_α in expression (7) for Π_α is replaced by the first adiabatic index $\Gamma_\alpha = \frac{\rho_\alpha p'_\alpha(\rho_\alpha)}{p_\alpha(\rho_\alpha)}$. The Petrovskii parabolicity of these systems follows from [9, 10].

Define the function $P_{0\alpha}(r) := \int_{r_0}^{r} \Big(\frac{r}{s} - 1\Big) p'_\alpha(s) ds$, where $r > 0$ and $r_0 > 0$. Let the body force density is such that $\mathbf{F}_\alpha(x, t) = \nabla \Phi_\alpha(x) + \mathbf{f}_\alpha(x, t)$, where $\nabla \Phi_\alpha$ is the density of the stationary potential body force and $\mathbf{f}_\alpha$ is its perturbation.

Theorem 2. *For the barotropic* QGD *system* (4), (5), *the following energy balance equation holds:*

$$\partial_t(E_{\text{pot}} + E_{\text{kin}}) + \operatorname{div}\mathbf{A} - \mathcal{P}_E = \sum_{\alpha = a,b} [\rho_\alpha - \tau \operatorname{div}(\rho_\alpha \mathbf{u}_\alpha)] \mathbf{f}_\alpha \cdot \mathbf{u}_\alpha + \rho_\alpha \mathbf{f}_\alpha \cdot \hat{\mathbf{w}}_{\Phi\alpha}, \tag{22}$$

where the potential and kinetic energies of the mixture have the form

$$E_{\text{pot}} = P_{0a}(\rho_a) - \rho_a \Phi_a + P_{0b}(\rho_b) - \rho_b \Phi_b,$$
$$E_{\text{kin}} = \frac{1}{2}(\rho_a |\mathbf{u}_a|^2 + \rho_b |\mathbf{u}_b|^2),$$
$$\mathbf{A} := \sum_{\alpha = a,b} \rho_\alpha (P'_{0\alpha}(\rho_\alpha) - \Phi_\alpha + 0.5|\mathbf{u}_\alpha|^2) \times (\mathbf{u}_\alpha - \mathbf{w}_\alpha) - \Pi_\alpha \mathbf{u}_\alpha,$$
$$\mathcal{P}_E := -\sum_{\alpha = a,b} \Big\{ \Xi_{NS,1\alpha} - \mathbf{S}_{u,\alpha} \cdot \mathbf{u}_\alpha + \frac{\rho_\alpha}{\tau}|\hat{\mathbf{w}}_{\Phi\alpha}|^2 + \tau \frac{p'_\alpha(\rho_\alpha)}{\rho_\alpha} [\operatorname{div}(\rho_\alpha \mathbf{u}_\alpha)]^2 \Big\} \tag{23}$$

is the energy production, and

$$\hat{\mathbf{w}}_{\Phi\alpha} = \frac{\tau}{\rho_\alpha} [\rho_\alpha (\mathbf{u}_\alpha \cdot \nabla)\mathbf{u}_\alpha + \nabla p_\alpha(\rho_\alpha) - \rho_\alpha \nabla \Phi_\alpha].$$

For $p'_\alpha(\rho_\alpha) \ge 0$, $\alpha = a, b$, *the energy production is nonpositive:* $\mathcal{P}_E \le 0$.

This result is proved by summing the energy balance equations (see [11, Proposition 1]) for the gases a and b and taking into account the exchange terms. The important inequality $\mathcal{P}_E \le 0$ follows from (12) and (17).

For the barotropic QGD system (19), (20), Theorem 2 remains valid with the only simplifications being that the term $-\tau\mathrm{div}(\rho_\alpha \mathbf{u}_\alpha)$ is omitted from the right-hand side of (22) and the last term is dropped from the energy production expression (23). Then $\mathcal{P}_E \le 0$ without imposing any conditions on $p'_\alpha(\rho_\alpha)$.

Note that all considered equations for binary mixtures, together with Theorems 1 and 2, can be directly extended to the case of $\tau = \tau_\alpha$.

ACKNOWLEDGMENTS

This work was supported by the Russian Science Foundation, project 14-11-00549.

REFERENCES

1. B. N. Chetverushkin, *Kinetic Schemes and Quasi-Gasdynamic System of Equations* (MAKS, Moscow, 2004; CIMNE, Barcelona, 2008).
2. T. G. Elizarova, *Quasi-Gas Dynamic Equations* (Nauchnyi Mir, Moscow, 2007; Springer-Verlag, Berlin, 2009).
3. Yu. V. Sheretov, *Continuum Dynamics under Spatiotemporal Averaging* (NITs RKhD, Moscow, 2009) [in Russian].
4. T. G. Elizarova, I. A. Graur, and J.-C. Lengrand, Eur. J. Mech. Ser. B **20** (3), 351–369 (2001).
5. Y. Wu and C. H. Lee, Phys. Fluids **14** (2), 313–322 (1971).
6. A. A. Zlotnik, Dokl. Math. **81** (2), 312–316 (2010).
7. A. A. Zlotnik, Mat. Model. **22** (7), 53–64 (2010).
8. R. Kamali, H. Emdad, and M. M. Alishahi, Fluid Dyn. Res. **40** (5), 343–363 (2008).
9. A. A. Zlotnik, Math. Notes **83** (5), 610–623 (2008).
10. A. A. Zlotnik, Comput. Math. Math. Phys. **50** (2), 310–321 (2010).
11. A. A. Zlotnik, Mat. Model. **24** (4), 65–79 (2012).

Translated by I. Ruzanova