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DISSIPATIVE TERMS IN QUASI-GAS-DYNAMIC EQUATIONS AND THEIR EFFECT ON SHOCK WAVE FLOW FIELD

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The general form of quasi-gas-dynamic equations has been presented, along with that of additional dissipative terms in cylindrical and rectangular Cartesian coordinates, and the effect of additional dissipation on gas flow parameters in a shock wave has been demonstrated.

The problem of describing gas flows with the aid of models extending the capabilities of the traditional Navier–Stokes (NS) equations has for a long time been of interest to investigators. One of such models is the system of quasi-gas-dynamic (QGD) equations [1–4]. This system describes the behavior of space-time averages, density, velocity, and temperature, rather than that of instantaneous space averages, as in the Navier–Stokes theory. The QGD equations differ from the NS equations by additional divergent terms with a time-dimension parameter as a coefficient. Additional dissipation present in the QGD equations makes efficient the numerical algorithms constructed on their base [1, 4, 5]. It is interesting to study the role of this additional dissipation, using, as an example, concrete gas-dynamic flows. Note that the QGD equations differ from other structurally close systems suggested in [6, 7].

The gas flow is described by means of three conservation laws, the law of conservation of mass, momentum, and energy, that may be represented in index form in ordinary terms as

$$\frac{\partial}{\partial t}\rho + \nabla_i J^i = 0, \tag{1}$$

$$\frac{\partial}{\partial t}(\rho\ u^k) + \nabla_i J^i u^k \ + \nabla^k p = \nabla_i\ \Pi^{ik}, \tag{2}$$

$$\frac{\partial}{\partial t}E + \nabla_i \frac{J^i}{\rho}(E+p) + \nabla_i q^i = \nabla_i(\Pi^{ik}u_k). \tag{3}$$

To close system (1)–(3), it is necessary to find the expressions for the mass flow density vector J^i, viscous stress tensor Π^{ik}, and heat flow vector q^i.

The system of Navier–Stokes equations describes the behavior of instantaneous space average values of the gas-dynamic parameters ρ, u^i, and p. Suitable expressions for the quantities J^i_{NS}, Π^{ik}_{NS}, and q^i_{NS} can be found, for example, in [8]. If one uses space-time averages to compute ρ, u^i, and p, then system (1)–(3) can be closed by two other methods [2, 3]. For an ideal polytropic gas, such a closure has the form

$$J^i_{QGD} = \rho u^i - \tau(\nabla_j(\rho u^i u^j) + \nabla^i p), \tag{4}$$

$$\Pi^{ik}_{QGD} = \Pi^{ik}_{NS} + \tau u^i(\rho u^j \nabla_j u^k + \nabla^k p) + \tau g^{ik}(u_j \nabla^j p + \gamma p \nabla_j u^j), \tag{5}$$

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$$q^i_{QGD} = -k\nabla^i T - \tau\rho u^i \left(u^j\nabla_j\varepsilon + pu_j\nabla^j\left(\frac{1}{\rho}\right)\right) \quad (6)$$

and forms a QGD system of equations. The second closure method gives rise to a QGD system valid for describing flows of imperfect gases and liquids.

In the above equations, ∇_i and ∇^i are the co- and contravariant derivatives, g^{ij} is the metric tensor, k is the thermal conductivity coefficient, $\varepsilon = p/(\rho(\gamma-1))$ is the specific internal energy, and γ is the adiabatic exponent. Expressions (4)–(6) include the relaxation parameter τ, which is of the same order of magnitude as the mean free time of molecules in a gas and can be related to the coefficient of viscosity of the latter by the relation $\tau \sim \mu/p$, where μ is the dynamic viscosity and μ/p is the Maxwellian relaxation time. Thus, two types of dissipative terms can be distinguished in the QGD equations, namely, the ordinary Navier–Stokes viscosity proportional to the coefficient of viscosity, μ, and an additional dissipation proportional to the coefficient τ.

Using the tensor analysis rules [9], we write the form of the QGD additions $\tilde{J}^i_{QGD}$, $\tilde{q}^i_{QGD}$, and $\tilde{\Pi}^{ik}_{QGD}$ to the mass flow density and heat flow vectors and to the viscous stress tensor in cylindrical and rectangular Cartesian coordinates.

In cylindrical coordinate system, an addition to the mass flow density vector in physical coordinates has the form

$$\tilde{J}_r^{\,QGD} = -\tau\left[\frac{1}{r}\frac{\partial}{\partial r}(r\rho u_r^2) + \frac{1}{r}\frac{\partial}{\partial\varphi}(\rho u_r u_\varphi) + \frac{\partial}{\partial z}(\rho u_r u_z) + \frac{\partial p}{\partial r}\right],$$

$$\tilde{J}_\varphi^{\,QGD} = -\tau\left[\frac{\partial}{\partial r}(\rho u_\varphi u_r) + \frac{1}{r}\frac{\partial}{\partial\varphi}(\rho u_\varphi^2) + \frac{\partial}{\partial z}(\rho u_\varphi u_z) + \frac{1}{r}\frac{\partial p}{\partial\varphi}\right],$$

$$\tilde{J}_z^{\,QGD} = -\tau\left[\frac{1}{r}\frac{\partial}{\partial r}(r\rho u_z u_r) + \frac{1}{r}\frac{\partial}{\partial\varphi}(\rho u_z u_\varphi) + \frac{\partial}{\partial z}(\rho u_z^2) + \frac{\partial p}{\partial z}\right].$$

An addition to the heat flow vector is expressed as follows:

$$\tilde{q}_r^{QGD} = -\tau\rho u_r\frac{1}{\gamma-1}\left[u_r\frac{\partial}{\partial r}\left(\frac{p}{\rho}\right) + \frac{u_\varphi}{r}\frac{\partial}{\partial\varphi}\left(\frac{p}{\rho}\right) + u_z\frac{\partial}{\partial z}\left(\frac{p}{\rho}\right)\right]$$
$$-\tau\rho u_r p\left[u_r\frac{\partial}{\partial r}\left(\frac{1}{\rho}\right) + \frac{u_\varphi}{r}\frac{\partial}{\partial\varphi}\left(\frac{1}{\rho}\right) + u_z\frac{\partial}{\partial z}\left(\frac{1}{\rho}\right)\right],$$

$$\tilde{q}_\varphi^{QGD} = -\tau\rho u_\varphi\frac{1}{\gamma-1}\left[u_r\frac{\partial}{\partial r}\left(\frac{p}{\rho}\right) + \frac{u_\varphi}{r}\frac{\partial}{\partial\varphi}\left(\frac{p}{\rho}\right) + u_z\frac{\partial}{\partial z}\left(\frac{p}{\rho}\right)\right]$$
$$-\tau\rho u_\varphi p\left[u_r\frac{\partial}{\partial r}\left(\frac{1}{\rho}\right) + \frac{u_\varphi}{r}\frac{\partial}{\partial\varphi}\left(\frac{1}{\rho}\right) + u_z\frac{\partial}{\partial z}\left(\frac{1}{\rho}\right)\right],$$

$$\tilde{q}_z^{QGD} = -\tau\rho u_z\frac{1}{\gamma-1}\left[u_r\frac{\partial}{\partial r}\left(\frac{p}{\rho}\right) + \frac{u_\varphi}{r}\frac{\partial}{\partial\varphi}\left(\frac{p}{\rho}\right) + u_z\frac{\partial}{\partial z}\left(\frac{p}{\rho}\right)\right]$$
$$-\tau\rho u_z p\left[u_r\frac{\partial}{\partial r}\left(\frac{1}{\rho}\right) + \frac{u_\varphi}{r}\frac{\partial}{\partial\varphi}\left(\frac{1}{\rho}\right) + u_z\frac{\partial}{\partial z}\left(\frac{1}{\rho}\right)\right].$$

Additions to the viscous stress tensor are of a fairly cumbersome form:

$$\tilde{\Pi}_{rr}^{QGD} = \tau\rho u_r\left[u_r\frac{\partial u_r}{\partial r} + \frac{u_\varphi}{r}\frac{\partial u_r}{\partial\varphi} - \frac{u_\varphi}{r} + u_z\frac{\partial u_r}{\partial z} + \frac{1}{\rho}\frac{\partial p}{\partial r}\right] + \tau\left[u_r\frac{\partial p}{\partial r} + \frac{u_\varphi}{r^3}\frac{\partial p}{\partial\varphi} + u_z\frac{\partial p}{\partial z} + \gamma p\,\mathrm{div}\,\mathbf{u}\right],$$

$$\tilde{\Pi}_{r\varphi}^{QGD} = \tau\rho u_r\frac{1}{r}\left[ru_r\frac{\partial u_\varphi}{\partial r} - r^2u_ru_\varphi + u_\varphi\frac{\partial u_\varphi}{\partial\varphi} + ru_z\frac{\partial u_\varphi}{\partial z} + \frac{1}{\rho}\frac{\partial p}{\partial\varphi}\right],$$

$$\tilde{\Pi}_{rz}^{QGD} = \tau\rho u_r\left[u_r\frac{\partial u_z}{\partial r} + \frac{u_\varphi}{r}\frac{\partial u_z}{\partial\varphi} + u_z\frac{\partial u_z}{\partial z} + \frac{1}{\rho}\frac{\partial p}{\partial z}\right],$$

$$\tilde{\Pi}_{\varphi r}^{QGD} = \tau\rho u_\varphi\left[u_r\frac{\partial u_r}{\partial r} + \frac{u_\varphi}{r}\frac{\partial u_r}{\partial\varphi} - \frac{u_\varphi^2}{r} + u_z\frac{\partial u_r}{\partial z} + \frac{1}{\rho}\frac{\partial p}{\partial r}\right],$$

$$\tilde{\Pi}^{QGD}_{\varphi\varphi} = \tau\rho u_\varphi \frac{1}{r}\left[r u_r \frac{\partial u_\varphi}{\partial r} + u_\varphi \frac{\partial u_\varphi}{\partial \varphi} + u_r u_\varphi + r u_z \frac{\partial u_\varphi}{\partial z} + \frac{1}{\rho}\frac{\partial p}{\partial \varphi}\right] + \tau\left[u_r \frac{\partial p}{\partial r} + \frac{u_\varphi}{r^3}\frac{\partial p}{\partial \varphi} + u_z \frac{\partial p}{\partial z} + \gamma p \operatorname{div}\mathbf{u}\right],$$

$$\tilde{\Pi}^{QGD}_{\varphi z} = \tau\rho u_\varphi \left[u_r \frac{\partial u_z}{\partial r} + \frac{u_\varphi}{r}\frac{\partial u_z}{\partial \varphi} + u_z \frac{\partial u_z}{\partial z} + \frac{1}{\rho}\frac{\partial p}{\partial z}\right],$$

$$\tilde{\Pi}^{QGD}_{zr} = \tau\rho u_z \left[u_r \frac{\partial u_r}{\partial r} + \frac{u_\varphi}{r}\frac{\partial u_r}{\partial \varphi} - \frac{u_\varphi^2}{r} + u_z \frac{\partial u_r}{\partial z} + \frac{1}{\rho}\frac{\partial p}{\partial r}\right],$$

$$\tilde{\Pi}^{QGD}_{z\varphi} = \tau\rho u_z \frac{1}{r}\left[r u_r \frac{\partial u_\varphi}{\partial r} + u_\varphi \frac{\partial u_\varphi}{\partial \varphi} + u_r u_\varphi + r u_z \frac{\partial u_\varphi}{\partial z} + \frac{1}{\rho}\frac{\partial p}{\partial \varphi}\right],$$

$$\tilde{\Pi}^{QGD}_{zz} = \tau\rho u_z \left[u_r \frac{\partial u_z}{\partial r} + \frac{u_\varphi}{r}\frac{\partial u_z}{\partial \varphi} + u_z \frac{\partial u_z}{\partial z} + \frac{1}{\rho}\frac{\partial p}{\partial z}\right] + \tau\left[u_r \frac{\partial p}{\partial r} + \frac{u_\varphi}{r}\frac{\partial p}{\partial \varphi} + u_z \frac{\partial p}{\partial z} + \gamma p \operatorname{div}\mathbf{u}\right].$$

Here

$$\operatorname{div}\mathbf{u} = \frac{1}{r}\frac{\partial (r u_r)}{\partial r} + \frac{1}{r}\frac{\partial u_\varphi}{\partial \varphi} + \frac{\partial u_z}{\partial z}.$$

Note that in contrast to the Navier–Stokes tensor, this additional tensor is asymmetric.

In rectangular Cartesian coordinates, the above expressions are simplified: an additional component of the mass flow density vector has the form

$$\tilde{J}_x^{\,QGD} = -\tau\left[\frac{\partial}{\partial x}(\rho u_x^2) + \frac{\partial}{\partial y}(\rho u_x u_y) + \frac{\partial}{\partial z}(\rho u_x u_z) + \frac{\partial p}{\partial x}\right],$$

an additional heat flow vector component is

$$\tilde{q_x}^{\,QGD} = -\tau\rho u_x \frac{1}{\gamma-1} \times \left[u_x \frac{\partial}{\partial x}\left(\frac{p}{\rho}\right) + u_y \frac{\partial}{\partial y}\left(\frac{p}{\rho}\right) + u_z \frac{\partial}{\partial z}\left(\frac{p}{\rho}\right)\right]$$
$$-\tau\rho u_x p\left[u_x \frac{\partial}{\partial x}\left(\frac{1}{\rho}\right) + u_y \frac{\partial}{\partial y}\left(\frac{1}{\rho}\right) + u_z \frac{\partial}{\partial z}\left(\frac{1}{\rho}\right)\right],$$

and the additions to the Navier–Stokes viscous stress tensor for diagonal elements are

$$\tilde{\Pi}^{QGD}_{xx} = \tau\rho u_x\left(u_x \frac{\partial u_x}{\partial x} + u_y \frac{\partial u_x}{\partial y} + u_z \frac{\partial u_x}{\partial z}\right) + \tau\left(2u_x \frac{\partial p}{\partial x} + u_y \frac{\partial p}{\partial y} + u_z \frac{\partial p}{\partial z}\right) + \tau\gamma p \operatorname{div}\mathbf{u},$$

and for nondiagonal elements,

$$\tilde{\Pi}^{QGD}_{xy} = \tau\rho u_x\left(u_x \frac{\partial u_y}{\partial x} + u_y \frac{\partial u_y}{\partial y} + u_z \frac{\partial u_y}{\partial z} + \frac{1}{\rho}\frac{\partial p}{\partial y}\right).$$

The cyclical permutation of coordinates ($x \to y,\ y \to z,\ z \to x$, and $u_x \to u_y,\ u_y \to u_z,\ u_z \to u_x$) helps one to obtain all the rest of the components.

The divergence in the above formulas is calculated as follows:

$$\operatorname{div}\mathbf{u} = \frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} + \frac{\partial u_z}{\partial z}.$$

Assuming that the flow is flat, unidimensional, and parallel to the Ox-axis, we consider a plane stationary shock wave whose front is perpendicular to the Ox-axis and located at the point $x = 0$. Such a flow is described by the system of QGD equations (1)–(3) in the form

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}\rho u = \frac{\partial}{\partial x}\tau\frac{\partial}{\partial x}(\rho u^2 + p),$$

$$\frac{\partial(\rho u)}{\partial t} + \frac{\partial p}{\partial x} + \frac{\partial}{\partial x}\left(\rho u^2 - (3-\gamma)\mu\frac{\partial u}{\partial x}\right) = \frac{\partial}{\partial x}\tau\left(\frac{\partial}{\partial x}(\rho u^3) + 3u\frac{\partial p}{\partial x} + \gamma p\frac{\partial u}{\partial x}\right),$$

$$\frac{\partial E}{\partial t} + \frac{\partial}{\partial x}\left(u(E+p) - \frac{\gamma \mathrm{Pr}^{-1}}{\gamma-1}\mu\frac{\partial}{\partial x}\left(\frac{p}{\rho}\right) - (3-\gamma)\mu u\frac{\partial u}{\partial x}\right)$$

$$= \frac{\partial}{\partial x}\tau\left(\frac{(E+p)}{\rho}\frac{\partial}{\partial x}(\rho u^2+p) + \frac{\rho u^2}{\gamma-1}\frac{\partial}{\partial x}\frac{p}{\rho} + \rho u^2 p\frac{\partial}{\partial x}\frac{1}{\rho}\right) + \frac{\partial}{\partial x}\tau\left(\rho u^3\frac{\partial u}{\partial x} + 2\rho u^2\frac{\partial p}{\partial x} + \gamma p u\frac{\partial u}{\partial x}\right). \quad (7)$$

Here the additional dissipative terms proportional to τ are placed on the right-hand side of the equations.

Taken as the boundary conditions at the right- and left-hand boundaries are the Rankine–Hugoniot conditions. In calculations, the quantities x, t, ρ, u, p, and E are made dimensionless by dividing into λ_1, λ_1/a_1, ρ_1, a_1, $\rho_1 a_1^2$, and $\rho_1 a_1^2$ (here λ is the mean free path of particles in the gas and $a = (\gamma p/\rho)^{1/2}$ is the sound velocity). The relationship between the coefficient of viscosity and the temperature is taken in the form $\mu \sim T^{\omega}$. The indices 1 and 2 correspond to the values of gas-dynamic quantities on the left and on the right of the wave front. The Mach number is calculated as $\mathrm{Ma} = u/a$. The calculations are carried out using a time-explicit difference scheme of second-order spatial accuracy. The steady-state solution of the initial-boundary problem is obtained as a limit of a time-evolving process. The algorithm used is similar to the one described in [4].

To reveal the role of the additional viscosity, we have run a series of computations wherein the parameter τ varied between 0 and $5\tau_0$, where $\tau_0 = \mu/p$. Reducing τ decreased the time step necessary for the scheme to be stable and increased the number of iterations to convergence. To show the convergence of the solution on a grid, the problem was solved on a series of grids with the number of their spatial nodes equal to 201, 601, and 1201.

Figures 1–4 present profiles of gas-dynamic quantities in the shock wave in the case of $\mathrm{Ma}_1 = 5$, $\mathrm{Pr} = 2/3$, $\omega = 0.5$, and $\gamma = 5/3$ (Pr is the Prandtl number). The gas density, temperature, and velocity in Figs. 1 and 2 are additionally normalized by means of the relations $\tilde{\rho} = (\rho-\rho_1)/(\rho_2-\rho_1)$, $\tilde{T} = (T-T_1)/(T_2-T_1)$, $\tilde{u} = (u-u_2)/(u_1-u_2)$. The solid line refers to the case $\tau = \tau_0$, the dashed line ($\tau = 0$), to the case of Navier–Stokes equations, and the dash-and-dot line corresponds to the case where the relaxation parameter is constant and calculated from the gas parameter values behind the shock wave ($\tau = \mu_2/p_2$). The line with circles corresponds to the solution of this problem by the Monte Carlo direct numerical modeling technique [4]. These data can be regarded as standard.

It follows from the above curves that additional dissipation has practically no effect on the width of the shock wave, which is calculated from the maximum slope of the density curve and is an important characteristic of the flow in a shock wave (Fig. 1). The flow velocity and viscous stress distributions also

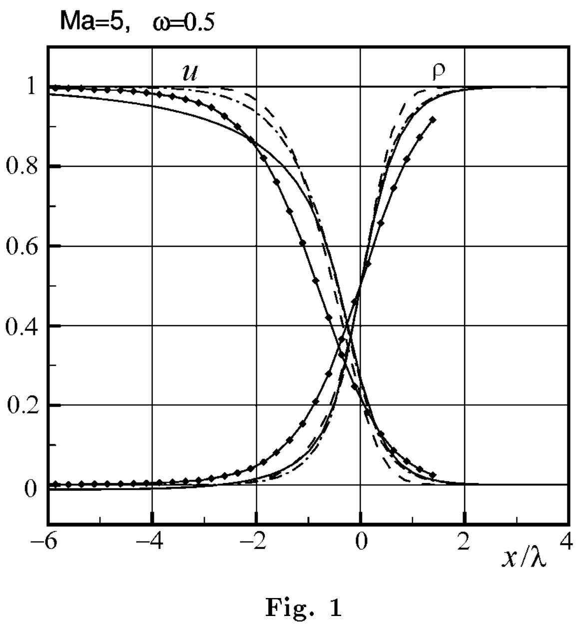

Fig. 1

Gas density and velocity profiles.

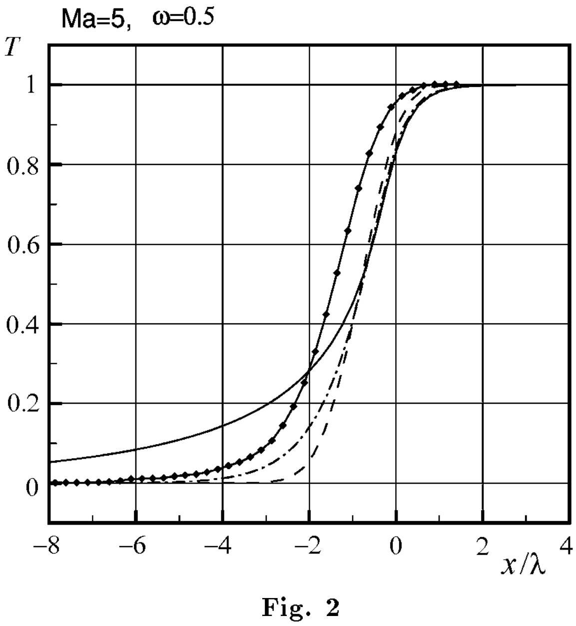

Fig. 2

Gas temperature profiles.

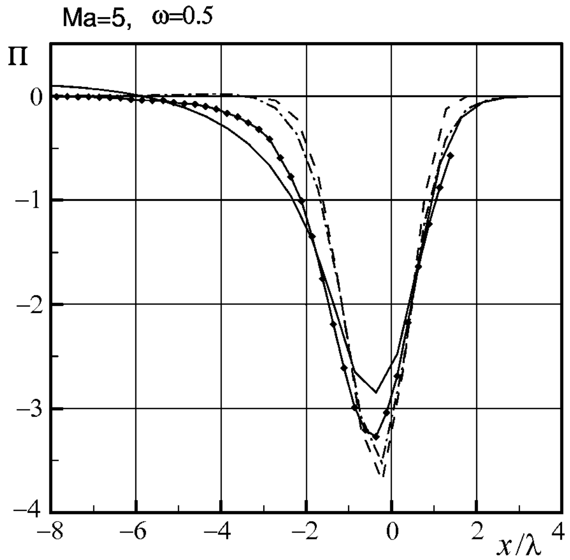

Fig. 3

Viscous stress profiles.

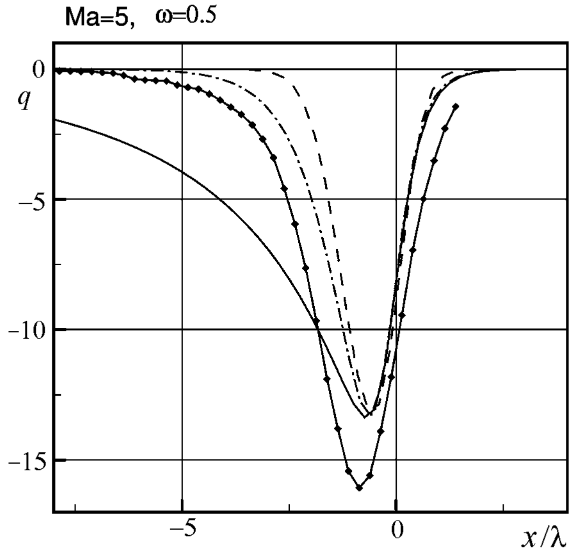

Fig. 4

Heat flow profiles.

depend but weakly on the value of the parameter τ (Figs. 1 and 3). The gas temperature (Fig. 2) and heat flow (Fig. 4*) prove to be most sensitive to the choice of the additional viscosity: putting $\tau = \tau_0$ leads to a noticeable increase in the gas temperature and heat flow in the region before the shock wave, where the mean free times of the particles are relatively long. The results close to the standard values are obtained for the computation version, wherein the relaxation parameter τ is selected to fit the flow parameters behind the shock wave. These results for all flow parameters exceed in accuracy the corresponding data obtained on the basis of the Navier–Stokes equations.

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* Note that the curve for q corresponding to the Navier–Stokes equations in our work [4] (Fig. 2) is erroneous.